J. Appl. Maths Mechs Vol. 57, No. 2, pp. 311–320, 1993
Printed in Great Britain.

0021–8928/93 $24.00+.00

OPTIMAL CONTROL OF THE PUMPING AND DAMPING OF A SWING†

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(*Received* 14 *January* 1992)

The problem of controlling, maximizing or minimizing the inclination of a swing to the vertical at its highest point is considered. The swing is modelled by a plane compound pendulum on which a point mass is located. This point can be displaced within limits along the straight line passing through the axis of support of the pendulum and its centre of mass. Viscous and dry friction are taken into account at the supporting axis, together with "aerodynamic" drag that is linear in the velocity and acting on the moving point. The controlling variable is the distance from the axis of support to the moveable point.

IT IS WELL KNOWN (see e.g. [1, 2]) that a swing with a person on it has its oscillations built up (pumped) if the centre of mass of the system is raised when it passes through its lowest point and lowered when it is at its highest point where its velocity is zero. Under certain conditions a similar control law, which will be explained in this paper, is optimal from the point of view of maximizing the deviation of the swing from the vertical at its highest point, i.e. at the end of an oscillation half-period. If, however, there is viscous friction due to air resistance and (or) there is dry friction at the hinge, the optimal control changes form. It turns out that the switch-over proceeds according to how the swing passes through its lowest point. Furthermore, this switch-over may proceed smoothly (continuously). An optimal swing-damping law is constructed which is in some sense the opposite of the pumping law.

The swing control problem is associated with the problem of using extendible rods to damp the oscillations of a satellite about its centre of mass in a gravitational field, and with control problems for some motions in sport. It is also of interest in theoretical mechanics.

1. THE EQUATIONS OF MOTION

We will model the swing–person system by a plane compound pendulum of mass m with a material point of mass M that can be displaced along it (Fig. 1). Let I be the moment of inertia of the pendulum about the point of support O and ρ the distance from the point O to the centre of mass of the pendulum C. The material point M can be moved along the line OC. Its distance OM from the point O is denoted by l. Let $l_0 \leq l \leq l_1$, where l_0, $l_1 = \text{const}$, $l_1 > l_0$.

The equation of motion of this system is

$$\frac{d}{dt}\left[(I + Ml^2)\frac{d\varphi}{dt}\right] + (cl^2 + c_1)\frac{d\varphi}{dt} + \nu + \zeta = 0, \qquad \zeta = (Ml + m\rho)g\sin\varphi \tag{1.1}$$

†*Prikl. Mat. Mekh.* Vol. 57, No. 2, pp. 92–101, 1993.

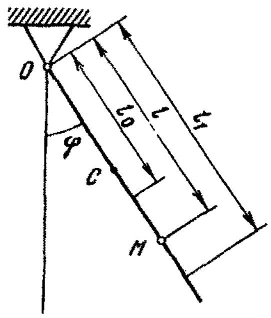

FIG. 1.

Here φ is the anticlockwise angle of deviation of the pendulum from the vertical (Fig. 1), $c>0$ is the coefficient of viscous damping due to air resistance opposing the motion (pumping) of the mass M (the person) [1], $c_1>0$ is the coefficient of viscous damping due to air resistance opposing the motion of the pendulum without the mass M (without the person), and also in the pivot, g is the acceleration due to gravity, and ν is the moment of the force of dry friction in the pivot, whose limit is equal to ν_0

$$\nu = \begin{cases} \nu_0 \operatorname{sgn}\dot\varphi, & \dot\varphi \neq 0 \\ -\zeta, & \dot\varphi = 0, \ |\zeta| \leqslant \nu_0 \\ \nu_0, & \dot\varphi = 0, \ \zeta \leqslant -\nu_0 \\ -\nu_0, & \dot\varphi = 0, \ \zeta \geqslant \nu_0 \end{cases} \tag{1.2}$$

We introduce dimensionless time τ, moment of inertia j, coefficients of viscosity χ and χ_1, the threshold of the moment of the frictional forces δ, the gravitational moment of the pendulum μ, and the distance OM denoted by u

$$\tau = \frac{t\sqrt{g}}{\sqrt{l_0}}, \quad j = \frac{I}{Ml_0^2}, \quad \chi = \frac{c\sqrt{l_0}}{M\sqrt{g}}, \quad \chi_1 = \frac{c_1}{Mgl_0\sqrt{l_0 g}}$$

$$\delta = \frac{\nu_0}{Mgl_0}, \quad \mu = \frac{m\rho}{Ml_0}, \quad u = \frac{l}{l_0}$$

Then Eq. (1.1) can be written in the form of a system of Cauchy equations

$$\begin{aligned} &\dot\varphi = K/(j+u^2) \\ &\dot K = -(\chi u^2 + \chi_1)K/(j+u^2) - \delta \operatorname{sgn} K - (u+\mu)\sin\varphi \end{aligned} \tag{1.3}$$

Here K is the total angular momentum of the system and the overdot denotes differentiation with respect to τ. The introduction of the phase variable K to replace $\dot\varphi$ enables us to avoid differentiating the variable u, which can change discontinuously, with respect to time. Together with $u(\tau)$ the velocity $\dot\varphi(\tau)$ undergoes discontinuities, but the angular momentum $K(\tau)$ remains a continuous function (so long as the viscous forces remain finite, which is the case in the model). The control u, together with the angle φ, occur non-linearly in system (1.3). Only the top line of (1.2) is used in system (1.3).

According to the above, the controlling parameter u is restricted

$$1 \leqslant u \leqslant U \qquad (U = l_1/l_0, \ U > 1) \tag{1.4}$$

We shall assume that $\delta < 1+\mu$. Then the stagnation zones of systems (1.1), (1.2)

$$-\varphi_*(u) \leqslant \varphi \leqslant \varphi_*(u), \qquad K=0 \tag{1.5}$$

$$\pi-\varphi_*(u) \leqslant \varphi \leqslant \pi+\varphi_*(u), \qquad K=0 \tag{1.6}$$

$$(\varphi_*(u)=\arcsin[\delta/(u+\mu)])$$

do not occupy the entire range $-\pi \leqslant \varphi < \pi$ when $u=1$, and the same is therefore true for any $u \in [1, U]$. Hence there are two intervals containing the points $\pi/2$ and $-\pi/2$ such that for angles $\varphi(0)$ contained in them dry friction cannot prevent the pendulum moving away from rest, whatever the position of the material point M.

2. STATEMENT OF THE PROBLEM

Let the initial state of system (1.3) lie outside the intervals (1.5), (1.6) for any value of $u \in [1, U]$

$$-\pi+\varphi_*(1) < \varphi(0) < -\varphi_*(1), \qquad K(0)=0 \tag{2.1}$$

We will formulate the problem of optimally controlled swing pumping: it is required to find a control law for the parameter u in the interval (1.4) for which max $\varphi(\theta)$ is obtained, where θ is the first instant after the zeroth instant of time when $K(\theta)=0$. We will write this formulation of the problem as follows:

$$\max_{1\leqslant u\leqslant U}[\varphi(\theta)], \qquad K(\theta)=0, \quad \theta>0 \tag{2.2}$$

We write the problem of the optimal damping law for the swing in the form

$$\min_{1\leqslant u\leqslant U}[\varphi(\theta)], \qquad K(\theta)=0, \quad \theta>0 \tag{2.3}$$

In the statement of the problem (2.2) (problem (2.3)) the initial state (2.1) is assumed to be such that a time θ exists at which the maximum (minimum) angle $\varphi(\theta)$ and $K(\theta)=0$ are reached.

Problems (2.2) and (2.3) are in essence Bulgakov's [3] problems for accumulated perturbations. The modified statement of the Bulgakov's problem, in which the time θ at the right-hand end is not specified, but is determined by a condition of the form $K(\theta)=0$, can be found in [4].

Under condition (2.1) the angular momentum $K>0$ in the time interval $0<\tau<\theta$. That is why only the upper line of relations (1.2) is used in Eqs (1.3).

3. METHOD OF SOLVING THE PROBLEM

For any admissible control $u(\tau)$ the angle $\varphi(\tau)$ increases strictly monotonically as τ increases from 0 to θ. Using this property when considering the phase trajectories of system (1.3) in the (φ, K) plane, one can verify that problem (2.2) is solved by a control u which maximizes the derivative $dK/d\varphi$ at each point of the trajectory. A control u which minimizes this derivative solves problem (2.3). The extremum of this derivative is reached at that value of $u \in [1, U]$ which extremizes the function

$$-\chi K u^2-[\delta+(u+\mu)\sin\varphi](j+u^2) \tag{3.1}$$

If $0 \leqslant \varphi \leqslant \pi$, the extrema of (3.1) are easily found: the maximum is reached at $u=1$ and the minimum at $u=U$. For $-\pi<\varphi<0$ an extremum of function (3.1) may be reached not only at boundary values $u=1$ or $u=U$, but at intermediate values $1<u<U$. If $-\pi<\varphi<0$ the maximum (minimum) of function (3.1) is reached at the same values of u as the maximum (minimum) of the function

$$F(u,A,j)=u(u^2+Au+j) \qquad \left(A=A(\varphi,K)=\frac{\delta+\chi K}{\sin\varphi}+\mu\right) \tag{3.2}$$

Below we shall use the equation

$$\partial F/\partial u=3u^2+2Au+j=0 \tag{3.3}$$

to find the extrema of function (3.2).

It follows from considerations of the functions (3.1) and (3.2) that the coefficient of viscosity χ_1, unlike χ and δ, has no effect on the picture of the optimal control synthesis.

We note the following property of the control which maximizes function (3.1), and the trajectory corresponding to it. Take any point (φ^*, K^*) lying on this trajectory. If for any other control the trajectory intersects the line $\varphi=\varphi^*$, then at the point of intersection $K \leqslant K^*$.

4. A SYSTEM WITHOUT AIR RESISTANCE OR DRY FRICTION

When there are no such resistance forces, i.e. when $\chi=\delta=0$, the maximum of function (3.1) is reached when $u=U$ if $-\pi<\varphi<0$, and at $u=1$ if $0\leqslant\varphi\leqslant\pi$. Considering the subsequent half-period of the swing oscillation (for $\tau>\theta$), we conclude that the optimal control has the form

$$u=U \quad \text{for} \quad \varphi K<0 \quad (\varphi\dot\varphi<0), \qquad u=1 \quad \text{for} \quad \varphi K\geqslant 0 \quad (\varphi\dot\varphi\geqslant 0) \tag{4.1}$$

The motion of massless swings ($j=\mu=0$) with control (4.1) was considered in [1], without, however, considering the question of its optimality.

The optimal damping law is "the opposite" of (4.1)

$$u=1 \quad \text{for} \quad \varphi K<0 \quad (\varphi\dot\varphi<0), \qquad u=U \quad \text{for} \quad \varphi K\geqslant 0 \quad (\varphi\dot\varphi\geqslant 0) \tag{4.2}$$

Control (4.1) maximizes the deviation of the swing from the vertical over any previously specified number of half-oscillations, whereas control (4.2) minimizes this deviation.

5. PUMPING OF THE SWING WHEN ALL FORCES ARE PRESENT

Taking into account all the dissipative forces described in system (1.3), the optimal control structure turns out to depend on the moment of inertia j. Analysis of function (3.2) and its derivatives with respect to u shows that three situations arise in the optimal control construction

$$(1)\ j\leqslant 2U+1,\ (2)\ 2U+1<j<3U^2,\ (3)\ 3U^2\leqslant j.$$

In case 1 function (3.1) reaches a maximum if

$$u=\begin{cases}U, & \delta+\chi K+\Lambda\sin\varphi\leqslant 0\\ 1, & \delta+\chi K+\Lambda\sin\varphi>0\end{cases} \qquad \left(\Lambda=U+\mu+\frac{j+1}{U+1}\right) \tag{5.1}$$

The switch-over line for control (5.1) is given by the equation

$$\delta + \chi K + \Lambda \sin \varphi = 0 \qquad (F(1, A, j) = F(U, A, j)) \tag{5.2}$$

Figure 2 shows the optimal control synthesis picture in the half-plane $K > 0$ for case 1 (the stagnation zone is shown hatched). The switch-over curve (5.2) intersects the semi-axis $K = 0$, $\varphi < 0$ at two points

$$\varphi = \varphi_0 = -\arcsin(\delta / \Lambda) \qquad \text{and} \qquad \varphi = -\pi - \varphi_0 \tag{5.3}$$

At $\chi = 0$ the curve (5.2) becomes the two lines (5.3).

The construction of the optimal control synthesis picture enables one to solve the optimization problem for more than just the initial states (2.1).

Control (5.1) is of relay form, as is (4.1). However, the switch-over of control (5.1) from $u = U$ to $u = 1$ occurs not at $\varphi = 0$, as for control (4.1), but earlier. As δ and χ increase and μ and j decrease, the switch-over point on each optimal trajectory moves away from the $\varphi = 0$ axis. The values of (5.3) lie in the intervals (1.5) and (1.6) obtained for $u = 1$. Hence for any initial state (2.1) at the beginning of the optimal motion $u = U$. If however $-\pi < \varphi(0) < -\pi - \varphi_0$, $K(0) > 0$, then at the beginning of the optimal motion $u = 1$.

When the coefficient of friction χ increases the switch-over time for control (5.1) from the value U to 1 approaches the start of the motion. This happens because, as χ increases, it becomes more favourable to reduce earlier the frictional torque $\chi u^2 K / (j + u^2)$ which opposes the pumping, despite the associated decrease of the pumping torque of the gravitational force $-u \sin \varphi$.

In case 2 the function (3.1) is maximized when

$$u = \begin{cases} U, & \delta + \chi K + \lambda(U) \sin \varphi \leqslant 0 \\ u_1(\varphi, K), & -\lambda(U) \sin \varphi < \delta + \chi K \leqslant -\eta(1) \sin \varphi \\ 1, & \delta + \chi K + \eta(1) \sin \varphi \geqslant 0 \end{cases} \tag{5.4}$$

$$u_1(\varphi, K) = -\frac{1}{3}(A + \sqrt{A^2 - 3j}), \quad \lambda(\upsilon) = 2\sqrt{j + \upsilon^2} - \upsilon + \mu, \quad \eta(\upsilon) = \frac{1}{2\upsilon}(j + 3\upsilon^2) + \mu$$

The quantity $u_1(\varphi, K)$ is the smaller of the two roots of quadratic equation (3.3).

Function (5.4) is discontinuous on the switch-over curve

$$\delta + \chi K + \lambda(U) \sin \varphi = 0 \qquad (F[u_1(\varphi, K), A, j] = F(U, A, j)) \tag{5.5}$$

Hence during optimal motion the control "descends" with a jump from the "buffer" U to the value $u_1(\varphi, K)$. Then, during the motion u decreases strictly monotonically and continuously to 1. At $j = 2U + 1$ the "relay-continuous" control (5.4) becomes the purely relay control (5.1).

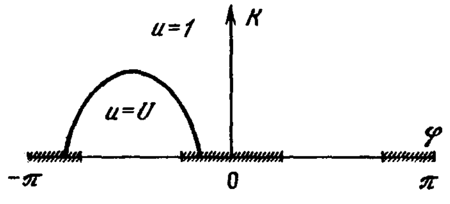

FIG. 2.

If $\chi = 0$, expression (5.4) becomes

$$u = \begin{cases} U, & \varphi \le \varphi_0 \\ u_1(\varphi, 0), & \varphi_0 < \varphi \le \varphi_1 \\ 1, & \varphi_1 \le \varphi \end{cases} \tag{5.6}$$

The angle φ_0 is obtained when solving Eq. (5.5), while the angle φ_1 is found by solving Eq. (3.1) with $u = 1$

$$\varphi_0 = -\arcsin[\delta / \lambda(U)], \qquad \varphi_1 = -\arcsin[\delta / \eta(1)] \tag{5.7}$$

The values of φ_0 and φ_1 obtained in (5.7) and the value of φ_0 in (5.3) are identical when $j = 2U + 1$.

As an example, Fig. 3 shows the graph of (5.6) for $\delta = 1$, $\mu = 0.5$, $j = 9$, $U = 2$ in the interval $[\varphi_0, \varphi_1]$ and in its neighbourhood.

In case 3 the function (3.1) is maximized under the control

$$u = \begin{cases} U, & \delta + \chi K + \eta(U)\sin\varphi \le 0 \\ u_1(\varphi, K), & -\eta(U)\sin\varphi \le \delta + \chi K \le -\eta(1)\sin\varphi \\ 1, & \delta + \chi K + \eta(1)\sin\varphi \ge 0 \end{cases} \tag{5.8}$$

Figure 4 shows the synthesis picture for optimal control (5.8) in the half-plane $K > 0$.

Unlike case 2, in case 3 the optimal control is continuous. After "descending" from the buffer U it decreases to unity strictly monotonically along each trajectory. At $j = 3U^2$ control (5.8) is identical with (5.4). If $\chi = 0$ the optimal control has the form (5.6). The value of φ_0 is obtained from Eq. (3.3) with $u = U$

$$\varphi_0 = -\arcsin[\delta/\eta(U)]$$

and is identical with that obtained in (5.7) with $j = 3U^2$.

The interval (φ_0, φ_1) is contained in the interval (1.5) obtained with $u = 1$. As δ increases and μ and j decrease, this interval moves away from the $\varphi = 0$ axis where the switch-over of control (4.1) occurs.

The results obtained here show that when there is dry friction at the pivot and (or) viscous friction due to air resistance to the motion of the mass M, in optimal motion the centre of mass of the system is displaced upwards not when the swing passes through its lowest point, as in the cases of control (4.1), but earlier.

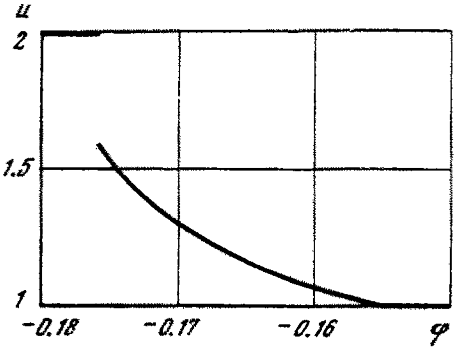

FIG. 3.

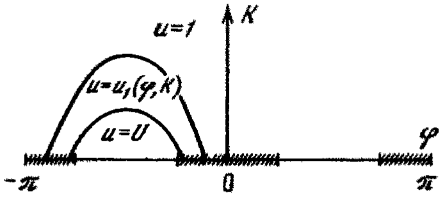

FIG. 4.

The optimal control synthesis picture for pumping is symmetric about the origin of coordinates. Hence it is easy to extend it to the half-plane $K < 0$.

It is of course not always possible to pump using optimal control. If, for example, the point $(\varphi(0), 0)$ is sufficiently close to the "lower" stagnation zone (1.5), then after a finite number of oscillations the system falls into this zone and the swing stops. Oscillations of the swing are also damped if $\delta = 0$ and the quantities χ and (or) χ_1 are sufficiently large. At the same time, for any initial angle $\varphi(0) \neq 0$ there are also sufficiently small values of χ, χ_1 and δ so that under optimal control the swing performs undamped oscillations. Initial conditions (2.1) and system parameters exist for which after one or several oscillations the swing "gets stuck" in the "upper" stagnation zone (1.6).

At a given initial state (2.1) we denote by φ' the maximum value of the angle $\varphi(\theta)$ at the time θ when $K(\theta) = 0$. Then the deflection φ' cannot be achieved faster than in the time θ. Consequently, controls (4.1), (5.1), (5.4) and also (5.8) are optimal in the sense of reaching the deviation φ' fastest.

We will assume that the system parameters and the initial state $\varphi(0)$, $K(0)$ are such that under a control (4.1), (5.1), (5.4) or (5.8) we have $\varphi(\tau) = \pi$, $K(\tau) > 0$ at some time τ, i.e. the swing "flies through" the upper equilibrium position. Then the corresponding control maximizes the total angular momentum $K(\tau)$ at the time τ when $\varphi(\tau) = \pi$. We have thus found a control, such that if its synthesis picture is constructed on a phase cylinder $-\pi \leq \varphi < \pi$, it solves not just the optimal pumping problem, but in a certain sense the optimal "revving-up" problem as well. It is clear that if $\chi \neq 0$, then for sufficiently large values of K the control $u = 1$ is optimal for revving-up. If $\chi = 0$, then, for example, in case 1 the switch-over curves for controlling revving-up are the lines (5.3). In the strip of the phase cylinder lying between these lines $u = U$, and outside the strip $u = 1$.

6. DAMPING OF A SWING

In taking account of all the forces described in system (1.3), the structure of the optimal control of swing damping, as for pumping, depends on the moment of inertia j. Analysis of the function (3.2) and its derivative with respect to u shows that the following three cases occur in the construction of this control

$$(1)\ j \leq 3,\ (2)\ 3 < j < U(U+2),\ (3)\ U(U+2) \leq j$$

i.e. the ranges of variation for j in which the structure of optimally controlled swing damping is uniform differ from the ranges in which the structure of optimally controlled swing pumping is uniform.

In case 1 function (3.1) is minimized if

$$u = \begin{cases} 1, & \delta + \chi K + \eta(1)\sin\varphi \leq 0 \\ u_2(\varphi, K), & -\eta(1)\sin\varphi \leq \delta + \chi K \leq -\eta(U)\sin\varphi \\ U, & \delta + \chi K + \eta(U)\sin\varphi \geq 0 \end{cases} \tag{6.1}$$

$$u_2(\varphi, K) = \frac{1}{3}(-A + \sqrt{A^2 - 3j})$$

The quantity $u_2(\varphi, K)$ is the largest of the two roots of the quadratic equation (3.3).

Control (6.1) is a continuous function of its arguments. On each trajectory, after "descent" from the buffer $u = 1$ its increases to the value U proceeds strictly monotonically.

The inequalities governing the domains where the control is not equal to its boundary values are opposite in formulae (5.8) and (6.1).

In case 2 (3.1) is minimized if

$$u=\begin{cases}1, & \delta+\chi K+\lambda(1)\sin\varphi\leqslant 0\\ u_2(\varphi,K), & -\lambda(1)\sin\varphi<\delta+\chi K\leqslant -\eta(U)\sin\varphi\\ U, & \delta+\chi K+\eta(U)\sin\varphi\geqslant 0\end{cases}\tag{6.2}$$

Function (6.2) becomes discontinuous at the switch-over curve

$$\delta+\chi K+\lambda(1)\sin\varphi=0 \qquad (F(1,A,j)=F[u_2(\varphi,K),A,j])$$

Hence in optimal motion the control u changes abruptly from the value 1 to $u_2(\varphi, K)$. Then it increases strictly monotonically and continuously to the value U.

In case 3 the function (3.1) is minimized if

$$u=\begin{cases}1, & \delta+\chi K+\Lambda\sin\varphi\leqslant 0\\ U, & \delta+\chi K+\Lambda\sin\varphi>0\end{cases}\tag{6.3}$$

The switch-over lines of (3.1) are described by Eqs (5.2).

The structure of the relay control (6.3) damping the swing is "opposite" to the structure of the relay control (5.1) pumping the swing. However condition 1 of Sec. 5, with which control (5.1) was obtained, is inconsistent with condition 3 of Sec. 6. With condition 1 of Sec. 5 the optimal control for swing damping can be of the form (6.1) or (6.2). Under condition 1 of Sec. 6 the optimal damping control is continuous, while the optimal pumping control (5.1) is relay. With condition 3 of Sec. 5 the pumping control (5.8) is continuous, while the damping control (6.3) is relay.

In all the damping control rules (6.1)–(6.3) the centre of gravity of the system is lowered not when the swing passes through the lowest point, as in the case of control (4.2), but earlier.

The swing pumping and damping problems posed in Sec. 2 are also solved if one assumes that not all values of u in interval (1.4) are admissible, but only the boundary values $u=1$ and $u=U$. Here the optimal control will always be relay, and its switch-over occurs on curve (5.2). If $\chi\neq 0$ and (or) $\delta\neq 0$, this switch-over occurs at $\varphi<0$.

7. PUMPING AND DAMPING OF A SWING IN A SPECIFIED TIME

We shall consider problems (2.2) and (2.3) for which the time θ has been specified in advance. These problems cannot be solved by extremizing function (3.1). Their solution is constructed by means of the Pontryagin maximum principle [5]. Here Eqs (1.3) must be supplemented by the following relations

$$\begin{aligned}
&\dot\psi_1=\psi_2(u+\mu)\cos\varphi, \qquad \dot\psi_2=[\psi_2(\chi u^2+\chi_1)-\psi_1\mp 1]/(j+u^2)\\
&H(u)=(\psi_1\pm 1)K/(j+u^2)-\psi_2[(u+\mu)\sin\varphi+\delta+(\chi u^2+\chi_1)K/(j+u^2)]\\
&\max_{1\leqslant u\leqslant U}H(u)=C, \quad u=\arg\Big[\max_{1\leqslant u\leqslant U}H(u)\Big]\\
&\psi_1(\theta)=0, \quad K(0)=K(\theta)=0
\end{aligned}\tag{7.1}$$

where $\psi_1(t)$ and $\psi_2(t)$ are conjugate variables and C is an unknown constant. The plus sign in the expression for $H(u)$ is for the pumping problem and the minus sign for the swing damping problem. Because we are considering motion in the half-plane $K>0$ we put $\operatorname{sgn}K=1$.

If the time θ is free, i.e. we are considering the problem posed in Sec. 2, then the optimal solution is described by relations (1.3) and (7.1) with $C=0$, from which we have $\psi_2(\theta)=0$. However, it is simpler to solve this case of the problem by finding extrema of function (3.1), as was done above.

If the time θ is specified, then $C \neq 0$. Setting the value of $\varphi(\theta)$ and the variable $\psi_2(\theta)$ at the final time θ and integrating relations (1.3) and (7.1) on a computer from right to left, one can construct a two-parameter family of extrema of the variational problem. To solve the optimality problem in question these parameters $\varphi(\theta)$ and $\psi_2(\theta)$ must be chosen so that the initial value of the angle φ is equal to the specified $\varphi(0)$ and the time to the specified θ.

From a consideration of relations (7.1) it follows that the optimal control regime can contain intervals of intermediate control. There are no intervals with singular control, where $\psi_1 \pm 1 \equiv 0$ and $\psi_2 \equiv 0$, because otherwise the boundary condition $\psi_1(\theta)=0$ will not be satisfied.

Figures 5 and 6 show two examples of some results of a numerical investigation of the swing pumping problem for $\delta=\chi=\chi_1=0$ (when there is no dissipation), $\mu=0.5$, $U=2$ and $\varphi(0)=-0.4248$. These investigations show that for negative values of $\psi_2(\theta)$ one obtains solutions with smaller time θ than when $\psi_2(\theta)=0$, while for positive values of $\psi_2(\theta)$ the time is greater. For $j=15$ ($j>3U^2$) the optimal control with the smaller time $\theta=8.58$ which is obtained for $\psi_2(\theta)=-0.2$ is shown on Fig. 5 by the solid curve, and with the larger time $\theta=10.38$, which is obtained for $\psi_2(\theta)=0.2$, by the dashed curve. This is the control with two switch-overs, continuous (the solid curve) and relay (the dashed curve). If the intervals with $u=1$ and $u=U$ are exchanged, the control shown by the dashed curve is qualitatively similar to the solid curve.

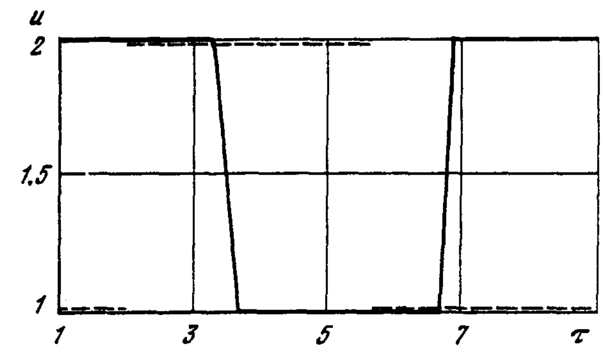

FIG. 5.

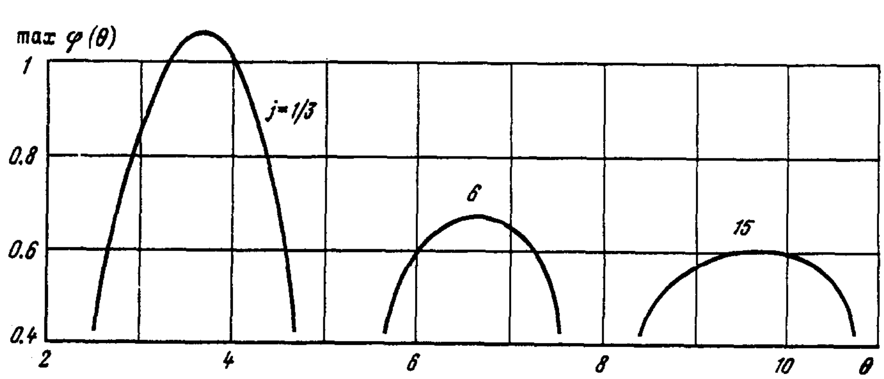

FIG. 6.

For the same pumping problem when there is dissipation in case 3 of Sec. 5 with the time θ being free, the optimal control (5.8) has one smooth switch-over. If a smaller time θ is specified and $\psi_2(\theta) \to -0$, the length of the last time interval in which $u > 1$ (solid curve) tends to zero, and the first interval where $u = 1$ increases, and the control tends to pure relay (4.1) with a free time θ which in the specified situation turns out to be 9.63. If the time θ is larger and $\psi_2(\theta) \to +0$, then the control tends to the same relay control (4.1), but here the length of the first interval, where $u < U$ (the dashed curve), tends to zero, while the length of the final interval, where $u = 1$, increases.

Numerical and analytical investigation of the pumping problem shows that here, when there is no dissipation, as well as in Sec. 5 where there is dissipation, cases 1, 2 and 3 appear, in which the optimal control takes less time than in Sec. 5, and which are of the pure relay, relay-continuous, and continuous form, respectively. Conversely, at larger times, the control is continuous, relay-continuous and pure relay, respectively. Unlike in Sec. 5, all the control regimes have two switch-overs.

Figure 6 shows the dependence of the maximum of the pumping amplitude $\varphi(\theta)$ on the specified time φ for a range of values of j. As one would expect, the maximum value of $\varphi(\theta)$ is obtained for a free time θ which is larger for smaller moments of inertia j. It is clear that for a given value of $\varphi(0)$ the first positive time θ at which $K(\theta) = 0$ cannot be chosen arbitrarily. It lies within certain limits. The numerical results shown in Fig. 6 illustrate this.

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Translated by R.L.Z.